Diversions

with Barry Squire

Sum fun with gnomons! or Take the Gausswork out of it!

Recall the story of schoolboy Gauss in the last issue? We pick up from there and use another neat way of adding lists of numbers to find a way of getting general formulae for figurate numbers and use Gauss's method to check it!

Returning to triangular numbers, to be able to talk about them, let's label them T_i for the ith triangular number. You might recall how the numbers build up (the gnomons were the counting numbers) so that each triangular number is the sum of so many counting numbers. Looking at a few:

$$T_3 = 1 + 2 + 3$$

Here's an odd thing to do; write it in reverse:

$$T_3 = 3 + 2 + 1$$

Add the two "equations" term by term:

$$2\,T_3 = 4 + 4 + 4$$

Write as a product rather than the answer:

$$2\,T_3 = 3 \times 4$$

so that

$$T_3 = \frac{(3 \times 4)}{2}$$

$$T_4 = 1 + 2 + 3 + 4$$

> But Gauss's method would be to add 1 + 3, 3/2 times, giving the same thing,
> $T_3 = \dfrac{(3 \times 4)}{2}$

Reverse:

$$T_4 = 4 + 3 + 2 + 1$$

Add termwise:

$$2\,T_4 = 5 + 5 + 5 + 5$$

Write as a product:

$$2\,T_4 = 4 \times 5$$

so that

$$T_4 = \frac{(4 \times 5)}{2}$$

> Gauss's method would be to add 1 + 4, 4/2 times, giving the same thing,
> $T_4 = \dfrac{(4 \times 5)}{2}$

$$T_5 = 1 + 2 + 3 + 4 + 5$$

Reverse:

$$T_5 = 5 + 4 + 3 + 2 + 1$$

Add termwise:

$$2\,T_5 = 6 + 6 + 6 + 6 + 6$$

Write as product:

$$2\,T_5 = 5 \times 6$$

so that

$$T_5 = \frac{(5 \times 6)}{2}$$

Gauss's method would be to add 1 + 5, 5/2 times, giving the same thing,
$T_5 = \frac{(5 \times 6)}{2}$

You can no doubt begin to see the pattern that leads to a generalised form:

$$T_n = 1+2+3+4+\ldots+n$$
$$= \frac{\left(n \times (n+1)\right)}{2}$$

(What would be Gauss's method?)

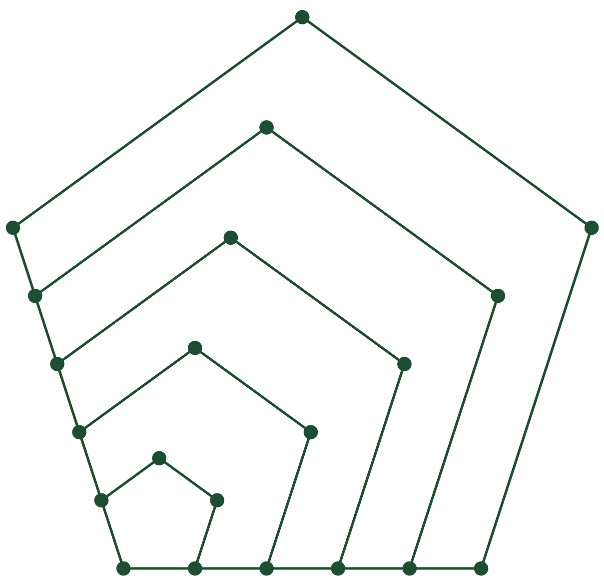
Test your theory with T_6, T_7, T_8, etc. Do you notice some vague connection of triangle to the "shape" of a rectangle or oblong?

The same technique can be applied to any other sequence of figurate numbers. Try it with the squares though you should be able to easily guess what S_n would be! Use it if you have time to verify the technique and check it using Gauss's method for

$Sn = 1 + 3 + 5 + \ldots + 2n–1$

(Why this *n*th term?)

Now we'll try it on the pentagonal numbers! We'll need to do some counting similarly to the triangular numbers but with a pentagonal grid of dots and develop a table of values that correspond to each "level" of number similar to the last edition of *Diversions* and identify the gnomon sequence.



Dots in row	1	2	3	4	5	6	7	8		N
Total dots in number	1	5	12							
Gnomon	4	7								

So, how do you get from one pentagonal number to the next one? What is the sequence of gnomons this time? Did you guess this sequence from your investigations last issue when you found the gnomon sequences for the triangular and square numbers?

We are now in a position to do the reversal addition trick to find the general form of these pentagonal numbers. Again we use P_3, P_4 and P_5 to see if we can see a pattern.

$$P_3 = 1 + 4 + 7$$

You should know this by now.

$$P_3 = 7 + 4 + 1$$

Reverse:

$$2\ P_3 = 8 + 8 + 8$$

Add:

$$2\ P_3 = 3 \times 8$$

so that

$$P3 = \frac{(3 \times 8)}{2}$$

Gauss check? 3/2 lots of 1+7.

$$P_4 = 1 + 4 + 7 + ?$$

Reverse:

$$P_4 = ? + 7 + 4 + 1$$

Add termwise

$$2\ P_4 = 11 + 11 + 11 + 11$$

(That's given it away now!)

$$2\,P_4 = 4 \times 11$$

so that

$$P_4 = \frac{(4 \times 11)}{2}$$

Gauss check? 4/2 lots of 1+10.

$$P_5 = 1 + 4 + 7 + ? + ??$$

Reverse:

$$P_5 = ?? + ? + 7 + 4 + 1$$

Add termwise:

$$2\,P_5 = \quad + \quad + \quad + \quad +$$

Write as product:

$$2\,P_5 = 5 \times ???$$

so that

$$P_5 = \frac{(5 \times ???)}{2}$$

Gauss check? 5/2 lots of 1+??

Can you begin to see the pattern that leads to a generalised form?

$$P_n = \frac{(n \times ?)}{2}$$

This ? is the key — the general way of writing the sequence for 5, 8, 11, 14...
The gnomon for this sequence is 3 isn't it?
Any help?
Step back:
gnomon for the odd numbers is 2, OK?
The general form for odd numbers is $2n - 1$.
The 2 is the key here — yes, the gnomon!

There are all sorts of ways of describing the process of deriving the general term for a sequence but whichever way you label it you are looking for a function of n (the position in the sequence) that gives you the actual term or value of the term in that position. It is a mapping of the counting numbers onto the sequence in question. Either way, it is a useful tool to have in your problem solving kit to develop your mathematical thinking with numbers!

If in the previous issue of *AMT* (page 15) you didn't try to test Fermat's declaration about figurate numbers being used to express any number, then you might be in a better position to do so now, having found a few more types of figurate numbers with which to play.

For example:

- 16 is a square number but also the sum of two triangular numbers, 6 and 10;
- 18 is the sum of two triangular numbers, 3 and 15.

So pick any number and find it to be a figurate number itself, or the sum of two or more of them. You might generate a table of these results and come to appreciate what a profound statement Fermat made over 350 years ago.

And just for fun, the Humble Pi riddle!

Nehemiah, a Hebrew scholar in the 2nd century AD gave the rule for finding the area of a circle:

> If you want to measure the area of a circle, multiply the diameter by itself and throw away from that one-seventh and half of one-seventh of it — the rest is the area.

What value of π does this suggest?

Solutions are on page 21.